A 2D Car Physics Model based on Ackermann Steering

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ABSTRACT

Car physics models are often complicated and require a large effort for understanding them. Additionally, some seem to be incomplete and many open questions remain. In this paper a novel 2D car physics model is presented. The model is based on Ackermann steering and presents closed formulas for forces causing rotation and acceleration of the car and tyres. As a simplification, the model assumes only two tyres instead of four. The paper describes how the equations are derived and how the model can be solved in a game loop.

Categories and Subject Descriptors

I.6 [**Simulation and Modeling**]: Miscellaneous

General Terms

Theory

Keywords

Car physics, Ackermann steering

1. INTRODUCTION

Car racing games rely on realistic car physics models for simulating the movement and traction of cars and tyres. Such a model may represent a car on a flat surface, as done in this work, or it may include the possibility of uneven ground and car jumps. Professional game studios put a large effort into developing proprietary car physics models for their products, which are of course not accessible for the public. People wishing to develop their own games thus have to rely on publicly available sources, which often are either over-sophisticated or leave open questions.

Alternatively, it is possible to use existing physics engines, for example general physics engines that allow to simulate rigid bodies and spring-mass systems, like ODE[1], Tokamak[2], Newton Game Dynamics[3], Havok[4], OPAL[5], or Bullet.[6] On the other hand, an engine specialized for the simulation of cars like Racer [12] can be used. In such a case, one is limited to the properties of the repective engine. Furthermore, using the engine might either be very costly, or the result might be restricted to non-commercial use due to legal considerations. Furthermore, the engine of course acts like a black box without insight into the model solution.

2. RELATED WORK

Basically, there are two types of car physics models found in the literature. The first one is a very sophisticated type of model which takes into account as many parts of a car as possible, like springs, suspensions, 3D landscape, tyre models and elaborate tyre slip models, etc. Implementing such a model requires a large amount of effort, its use is for instance in the car industry or for professional games [6, 11, 9, 21, 2].

The second type of papers describes very basic 2D models, here mainly taking into consideration engine and centripetal forces [7, 8, 10, 3, 4]. These models are easily understood, yet one has the impression that important parts of the models are missing. Examples for such gaps include breaking forces due to steering, adding rotational forces to the tyre traction budget, or the fact that the engine/break force not only accelerates the car mass (which includes the tyre masses), but parts of it also must cause the tyres to rotate. These issues will be treated later in the paper. Also, their integration and solution is often not obvious.

The aim of this paper is to focus on the second type of publication, i.e., an easily to be understood 2D model, which however tries to catch all forces of a car driving through a 2D plane. The presented model is based on so-called Ackermann steering, and consists of closed formulas, thus allowing to gain insight into the model behavior and solution. Furthermore, the integration and solution of the presented formulas in a game loop is presented.

3. RIGID BODY DYNAMICS

In this section those principles of rigid body dynamics which are necessary for understanding the proposed model will be

[1]http://www.ode.org/
[2]http://www.tokamakphysics.com/
[3]http://www.physicsengine.com/
[4]http://www.havok.com/
[5]http://ox.slug.louisville.edu/∼o0lozi01/opal_wiki
[6]http://www.continuousphysics.com/Bullet/

described briefly. For more detailed introductions in this field refer to [7, 8, 4, 5] or various articles at Wikipedia [17].

In the following, the discussion is restricted to the dynamics of solid cuboids, which are used in this paper to represent car masses. As a further simplification, we assume that the cuboid lies flat on a 2D plane, and its movements are restricted to the movements inside this 2D plane, just like a car may roll on the flat surface of a street.

Note that in the following, scalars and scalar operations instead of vectors and vector operations will be used *wherever possible*, since the forces observed are often at right angles to the main model axes. If used, vectors are explicitly denoted by using bold fonts. Also note that throughout the model, forces and accelerations are kept constant over a short amount of time $dt > 0$. In fact, dt is assumed to be so small that $\sin dt \approx dt$. This can always be achieved by letting $dt \to 0$, since it can easily be shown that [18]

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

for instance by using the Taylor series of $\sin x$ [20].

3.1 Translation

Assume a cuboid of length L, width W and height H meters, and having a mass of M kilograms. The height however will not be used in the remaider of this paper, its significance is only given in some cases when the mass of the cuboid has to be calculated. Assume also that we observe this cuboid from above, thus not seeing its heigth dimension. Additionally, the car at time t is at position $(x(t), y(t))$ in the plane and is moving upwards in positive y direction with *constant velocity* v. This kind of movement is called *translation*. If no force is applied to such an object, then the velocity will not change, and after a time dt has passed, the car's change of position dy in the y direction will be

$$dy = y(t + dt) - y(t) = v\, dt. \tag{1}$$

No matter what velocity $v(t)$ the cuboid is travelling at a time t, when we apply a *constant force* F to the center of its mass C (see Fig. 1), $v(t)$ is no more constant, and after time dt the total change of velocity is given by

$$dv = v(t + dt) - v(t) = \frac{F}{M} dt = a\, dt. \tag{2}$$

The factor $a = F/M = dv/dt$ is called the *acceleration*

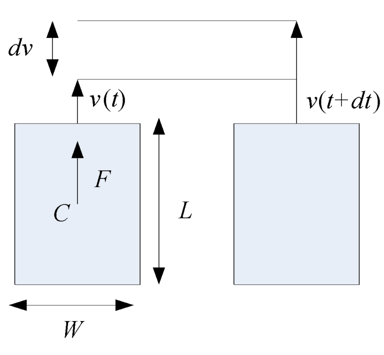

Figure 1: A cuboid is accelerated by a force F.

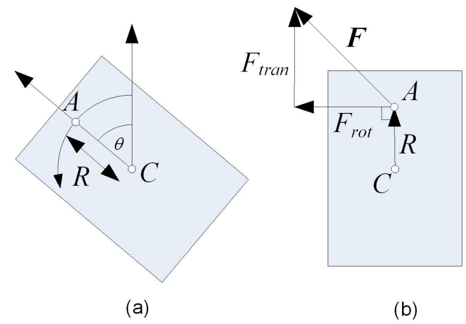

Figure 2: A cuboid rotating about an axis through its center (a). A force F is applied to a point A (b).

and represents the change of velocity per time unit, i.e., the first derivative of the velocity. Note that generally v and a are either both two-dimensional vectors, or scalars, if the direction is fixed, and only the magnitude is of importance. Also note that in the above scenario, the change in y-position dy of the car after dt is

$$dy = \bar{v}(t, dt)\ dt = v(t)\, dt + \frac{a}{2}\, dt^2, \tag{3}$$

due to the fact that the mean velocity $\bar{v}(t, dt)$ of the cuboid in the time interval $[t, t + dt]$ is given by

$$\bar{v}(t, dt) = \frac{v(t) + v(t + dt)}{2} = \frac{v(t) + v(t) + a\, dt}{2}, \tag{4}$$

i.e., the mean of $v(t)$ and $v(t + dt)$.

3.2 Rotation

The second important movement a cuboid may carry out in the plane is *rotating* about an axis. In this work we always assume that the rotation axis is perpendicular to the plane. Fig. 2 (a) shows a cuboid rotating about an axis going through its center C. Like in the case for translation, we can define a current state of the object at time t, which is given by the orientation angle $\theta(t)$. Additionally, the change of this angle, i.e., the angular velocity, is given by $\omega = d\theta/dt$, and angular acceleration is defined to be $\alpha = d\omega/dt$. Due to the rotational movement, a sample point A being at a distance of R away from the axis, moves along a circle with radius R. Its velocity v_R on this circle is given by

$$v_R = \omega R \tag{5}$$

and likewise its acceleration by

$$a_R = \alpha R. \tag{6}$$

Acceleration is related to a force $\boldsymbol{F}$ being applied to a point of the cuboid. Fig. 2 (b) shows the case where a force $\boldsymbol{F}$ is applied to a point A. Here, the force vector $\boldsymbol{F} = (F_{rot}, F_{tran})^T$ must be split into two components. The first component F_{rot} is *perpendicular* to the vector $\overrightarrow{\boldsymbol{CA}}$ and results in a rotational force changing ω. The remaining component F_{tran} is simply a translational force and must be treated as described in (1) to (4). F_{rot} then results in a *torque* [15]

$$T = F_{rot} R. \tag{7}$$

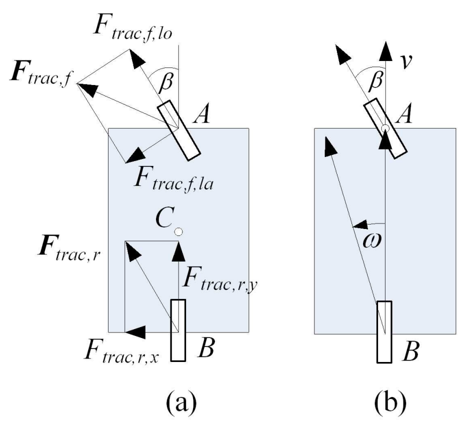

Figure 3: The base car model with traction forces (a) and the car movement model (b).

Similar to (2), this torque is then translated into angular acceleration by

$$\alpha = \frac{T}{I}, \tag{8}$$

where I is the rotational equivalent to mass called *moment of inertia*. For general shapes and axes, I must be described by means of so-called *tensors*. However, for a cuboid and an axis going through the center of mass C and being perpendicular to the upper cuboid face, I_C is defined to be [14]

$$I_C = \frac{M(W^2 + L^2)}{12}. \tag{9}$$

An important result is given by the *parallel axes theorem* due to J. Steiner, which describes the moment of inertia I if the axis is parallel to the one through C, and having a distance of R from it [19]:

$$I = I_C + MR^2. \tag{10}$$

4. A 2D CAR PHYSICS MODEL

In the proposed model the car is modelled by a cuboid as described in the previous section, with length L, width W and mass M. As convention we always assume that the car is heading up, i. e., its orientation is along the positive y-axis.

Furthermore, as a simplification, the car uses only two tyres instead of four, the tyres being at the middle of the front and rear sides, depicted in Fig. 3 (a) as points A and B. At these points, the major forces are exchanged between the car body, the tyres, the engine and the ground. The figure also shows the traction force

$$\begin{aligned} \boldsymbol{F}_{trac,f} &= \begin{pmatrix} F_{trac,f,x} \\ F_{trac,f,y} \end{pmatrix} \\ &= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} F_{trac,f,la} \\ F_{trac,f,lo} \end{pmatrix} \end{aligned} \tag{11}$$

at the front tyre, and

$$\boldsymbol{F}_{trac,r} = \begin{pmatrix} F_{trac,r,x} \\ F_{trac,r,y} \end{pmatrix} \tag{12}$$

at the rear tyre. Both forces can be split into two components. The first is a *longitudinal* component ($F_{trac,f,lo}$ and $F_{trac,r,y}$), and points into the direction of the respective tyre. This component is mainly responsible for accelerating the car. The second component ($F_{trac,f,la}$ and $F_{trac,r,x}$) is the *lateral* force, which is responsible for keeping the car on the current track.

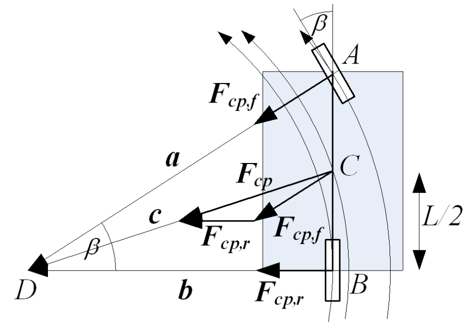

Figure 4: Ackermann steering and centripetal force.

Steering is modelled by the steering angle β, which defines the current angle between the front tyre and the main car direction (see Fig. 3 (b)). The current velocity of the car is represented by v. Furthermore, due to steering, the car may rotate by an angular velocity ω, the rotational axis here going through point B. However, as later shown in (23), ω can be computed by β, the current longitudinal velocity v, and L.

4.1 Limitations of the Model

The described model is kept simple on purpose and does not implement a number of features. The omitted features, however, are mostly orthogonal to the model and can be added to the equations if so desired.

First, the model does not implement weight transfer, which happens when the car is accelerated [10, 2]. Second, the model assumes an engine or brake torque to be fed from the engine into the tyres. Elaborated models for computing the engine torque as a function of gear and rpm exist, and can easily be used in addition to this model [21, 10]. Third, the model assumes only two tyres instead of four, thus blurring the possibility of different forces on the two front or rear tyres. Fourth, there is no slip between tyres and the ground, i.e., the slip ratio is set to 1 (see Section 4.6). As a consequence, there are also no applications of Pacejka's Magic Formula, which models properties of the tyre and the resulting slip [11]. Finally, the presented model does not contain aerodynamic drag and rolling resistance, which impose a breaking force onto a moving car [10].

The major source for driving a car is of course its engine. However, once moving, different forces appear and either move the car to the side or reduce its speed. In the following, three major types of force are investigated and described by equations. The integration of these equations into one closed model is then presented in Section 6.

4.2 Ackermann Steering

For cars there exists a movement model called *Ackermann steering* [6] (Fig. 4). The car center C moves along a circle,

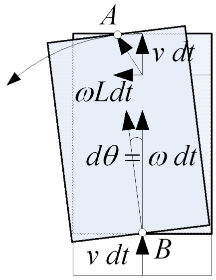

Figure 5: Movement model.

the rear tyre moves along a circle being concentric to the first one, but with smaller radius, and the front tyre moves along another concentric circle with larger radius. The center D of the three circles is the intersection of the vectors $\boldsymbol{a} = \overrightarrow{AD} = (a_x, a_y)^T$ and $\boldsymbol{b} = \overrightarrow{BD} = (b_x, 0)^T$, which originate at the tyre centers and which are perpendicular to the tyre orientations. If $\beta = 0$ then the two vectors are parallel and meet at infinity. Note that for $\beta \neq 0$ the center of mass C circumvents a larger circle than the one B circumvents in the same time, its velocity $\boldsymbol{v}_C$ thus must be larger than the velocity $|\boldsymbol{v}_B|$ of B, the same is true for points A and C. For $\beta \neq 0$ it follows that

$$|\boldsymbol{v}_B| < |\boldsymbol{v}_C| < |\boldsymbol{v}_A|. \tag{13}$$

In the following, these properties are the only premises for the car physics model described in this paper.

Simple calculation shows that the angle in the left corner of the triangle BDA is also the steering angle β. For given L, it follows that for $\beta \neq 0$ and $|\beta| < \pi/2$

$$b_x = -L \ \cot\beta \tag{14}$$

and

$$\boldsymbol{a} = \begin{pmatrix} b_x \\ -L \end{pmatrix} = \begin{pmatrix} -L\cot\beta \\ -L \end{pmatrix}. \tag{15}$$

Similarly, the vector $\boldsymbol{c} = \overrightarrow{CD}$ connecting the center of mass C with the circle center D is given by

$$\boldsymbol{c} = \begin{pmatrix} b_x \\ -L/2 \end{pmatrix} = \begin{pmatrix} -L\cot\beta \\ -L/2 \end{pmatrix}. \tag{16}$$

In the presented model, the car is moved in the following way (see Fig. 5). In each time period dt, the car first moves a length of $ds = v\,dt$ into the direction of the car, then the car is rotated by an angle of $d\theta = \omega\,dt$, using B as rotational axis. At this point it must be noted that it is thinkable that the rotational axis might go through the center or mass C, an assumption being found in many other 2D models. However, the following argument does not support this assumption. Suppose the rotational axis goes through C and not through B. The velocity of point C then is given by $\boldsymbol{v}_C$, point B would also have this velocity component, but additionally, due to the rotation around C, a second velocity component being perpendicular to $\boldsymbol{v}_C$. For $\beta \neq 0$ it follows that $|\boldsymbol{v}_B| > |\boldsymbol{v}_C|$. This, however is in contradiction to (13).

Due to the rotation, A additionally moves along a circle with speed ωL. As a consequence, point B moves with velocity vector $\boldsymbol{v}_B = (0, v)^T$, whereas point A moves with velocity $\boldsymbol{v}_A = (\omega L, v)^T$, and point C with velocity $\boldsymbol{v}_C = (\omega L/2, v)^T$, which is consistent with (13). If the car moves along a circle because of steering, this means that the points move at different circles being concentric, but having different radii. A travels along the largest circle, while C travels along a smaller circle, and B again on a smaller circle, exactly as defined in the Ackermann steering model.

Because of (14), (15), and (16) the relations between the different velocities are given by

$$\frac{|\boldsymbol{c}|}{|\boldsymbol{b}|} = \frac{|\boldsymbol{v}_C|}{|\boldsymbol{v}_B|} = \frac{|\boldsymbol{v}_C|}{v} = \sqrt{1 + \frac{\tan^2\beta}{4}} \geq 1, \tag{17}$$

and

$$\frac{|\boldsymbol{b}|}{|\boldsymbol{a}|} = \frac{|\boldsymbol{v}_B|}{|\boldsymbol{v}_A|} = \frac{v}{|\boldsymbol{v}_A|} = \cos\beta \leq 1. \tag{18}$$

4.3 Centripetal Force

If the steering angle β is different to zero, then a lateral force is put onto the front tyre, pushing the front of the car to the respective side. For a fixed β, the car then travels along a circle. If an object with mass M moves with velocity v along a circle with radius R, then there must be a *centripetal force* $\boldsymbol{F}_{cp}$ pushing the object center C to the circle center. The length of $\boldsymbol{F}_{cp}$ is known to be [13] (see Fig. 4)

$$|\boldsymbol{F}_{cp}| = \frac{M\,|\boldsymbol{v}_C|^2}{R}. \tag{19}$$

From (16), (17) and (19), and noting that in this case $R = |\boldsymbol{c}|$, we get

$$\begin{aligned} \boldsymbol{F}_{cp} &= \frac{M\,(1 + \frac{\tan^2\beta}{4})\,v^2}{|\boldsymbol{c}|}\,\frac{\boldsymbol{c}}{|\boldsymbol{c}|} \\ &= \frac{-M\,(1 + \frac{\tan^2\beta}{4})\,v^2}{L(\cot^2\beta + 1/4)} \begin{pmatrix} \cot\beta \\ 1/2 \end{pmatrix} \\ &= \frac{-Mv^2\tan^2\beta}{L} \begin{pmatrix} \cot\beta \\ 1/2 \end{pmatrix} \end{aligned} \tag{20}$$

This is the centripetal force pulling the car towards the center of the circle it runs on. Also, it must be poduced by the lateral (cornering) forces of the front and rear tyres which are parts of the overal traction forces. Thus the next step is to split $\boldsymbol{F}_{cp} = \boldsymbol{F}_{cp,f} + \boldsymbol{F}_{cp,r}$ into two cornering forces originating at the front ($\boldsymbol{F}_{cp,f}$) and rear tyres ($\boldsymbol{F}_{cp,r}$), and pointing along the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ towards D. Since the forces responsible for the car *rotation* are modelled in Sections 4.4 and 4.5, the forces $\boldsymbol{F}_{cp,f}$ and $\boldsymbol{F}_{cp,r}$ treated here do not change the car's rotation. From the description of rotational forces in Section 3 it follows that the force components which are *perpendicular* to the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$ must be equal. When the car points up these components are the x-coordinates of $\boldsymbol{F}_{cp,f}$ and $\boldsymbol{F}_{cp,r}$, which therefore must be equal:

$$\boldsymbol{F}_{cp,f} = \frac{-Mv^2\tan^2\beta}{L} \begin{pmatrix} \frac{\cot\beta}{2} \\ 1/2 \end{pmatrix}, \tag{21}$$

and

$$\boldsymbol{F}_{cp,r} = \frac{-Mv^2\tan^2\beta}{L} \begin{pmatrix} \frac{\cot\beta}{2} \\ 0 \end{pmatrix}. \tag{22}$$

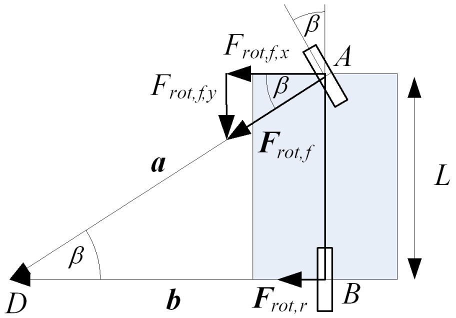

Figure 6: Forces causing the car to rotate.

It is worth noting that the centripetal forces $\boldsymbol{F}_{cp,r}$ and $\boldsymbol{F}_{cp,f}$ depend on the current steering angle β and the current velocity v, but not on the steering change $d\beta/dt$. Also note that both forces approximate the zero vector for $\beta \rightarrow 0$.

4.4 Car Rotation

A car travelling along a circle will also rotate as described in Section 3.2. Assume that the car is travelling with velocity v around a circle with radius R and circumference $l = 2R\pi$. Since for a complete run around the circle it needs the time $dt = l/v$ and rotates around an angle $d\theta = 2\pi$, the rotation is carried out with angular speed $\omega = d\theta/dt$. Since v is the velocity of point B we set $R = -b_x$ and derive because of (14)

$$\omega = \frac{d\theta}{dt} = \frac{2\pi}{l/v} = \frac{v}{-b_x} = \frac{v}{L\cot\beta} = \frac{v}{L}\ \tan\beta. \qquad (23)$$

Consider the car in Fig. 6. The force responsible for the rotation is $\boldsymbol{F}_{rot,f} = (F_{rot,f,x}, F_{rot,f,y})^T$ caused by the front tyre. To be exact, only the x-coordinate $F_{rot,f,x}$ causes the rotation, the other component $F_{rot,f,y}$ acts as a breaking force.

Since we assume that the car's rotation axis goes through its rear tyre, from (10) we know that the car's moment of inertia is $I_B = I_C + M\,L^2/4$. Thus, if the velocity v or the steering angle β change within a time dt by the amount of dv or $d\beta$, then from (23) we get [16]

$$\alpha = \frac{d\omega}{dt} = \frac{\tan\beta}{L}\frac{dv}{dt} + \frac{v}{L\ \cos^2\beta}\frac{d\beta}{dt}. \qquad (24)$$

Following (7) and (8), the force component $F_{rot,f,x}$ of the front tyre must be

$$F_{rot,f,x} = -\frac{\alpha\, I_B}{L} = -\frac{I_B}{L^2}\left(\tan\beta\frac{dv}{dt} + \frac{v}{\cos^2\beta}\frac{d\beta}{dt}\right), \qquad (25)$$

and furthermore

$$\begin{aligned} F_{rot,f,y} &= F_{rot,f,x}\ \tan\beta \\ &= -\frac{I_B\ \tan\beta}{L^2}\left(\tan\beta\frac{dv}{dt} + \frac{v}{\cos^2\beta}\frac{d\beta}{dt}\right). \qquad (26) \end{aligned}$$

Another aspect of rotation is the fact that the car rotates around an axis going through point B. This rotation also demands a centripetal force $\boldsymbol{F}_{cp,B}$, pushing the car against B, since without this force the car would rotate around its center of mass C. As the centripetal force always points to the rotational axis, the only point where such a force might be created is the front tyre. From there, it must point straight down to the rear tyre, i.e., its x-coordinate must be zero. Similar to (19), and by using (23), the y-coordinate of this force then must be

$$F_{cp,B,y} = -\frac{M\,(\omega L/2)^2}{L/2} = -\frac{Mv^2\tan^2\beta}{2L},$$

which is the same as the y-coordinate $F_{cp,f,y}$ of the centripetal force at the front tyre $\boldsymbol{F}_{cp,f}$ as given by (21). In fact, this argument proves that the same force $\boldsymbol{F}_{cp,f}$ that is responsible for keeping the car on its track at the front tyre also is responsible for letting the car rotate around its rear tyre, and not its center of mass.

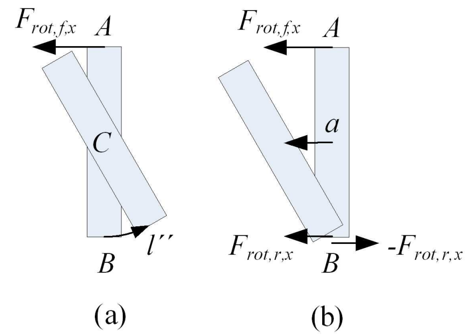

Figure 7: A force causing a cuboid to rotate (a) and application of two forces additionally causing an acceleration a (b).

4.5 A Mysterious Force

At this point it must be noted that there is also a lateral force at the rear tyre responsible for the rotation, though this force may not be visible at first sight. Think of a force causing a cuboid to spin. If only one force is applied, then the cuboid must rotate around its center of mass C. However, in the presented model, the rotational axis is the rear tyre, not the center of mass. It follows that there must be a second force, this time a lateral force $\boldsymbol{F}_{rot,r} = (F_{rot,r,x}, 0)^T$ at the rear tyre, which shifts the rotational axis to the rear.

Consider the scenario depicted in Fig. 7 (a). A force $F_{rot,f,x}$ induces an angular acceleration

$$\alpha = \frac{F_{rot,f,x}L/2}{I_C}$$

which in turn will cause the cuboid to rotate about the axis going through its center of mass C. The angular acceleration also causes an acceleration $l'' = \alpha L/2$ of point B.

Now consider Fig. 7 (b). Here, B is fixed to an axis and $F_{rot,f,x}$ causes some $F_{rot,r,x}$ to press against this axis, causing an equal force into the opposite direction. The result is that the force causing rotation now is only $F_{rot,f,x} - F_{rot,r,x}$:

$$\alpha = \frac{(F_{rot,f,x} - F_{rot,r,x})L/2}{I_C}$$

and the acceleration of B is given by

$$l'' = \frac{(F_{rot,f,x} - F_{rot,r,x})\ L^2/4}{I_C}.$$

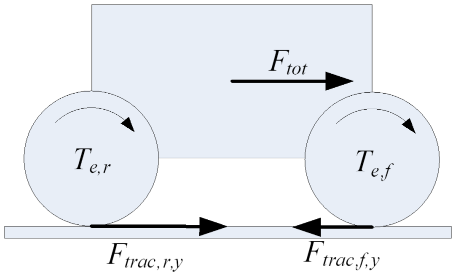

Figure 8: Acceleration forces at the tyres and the car body.

Additionally, the center of mass C is accelerated by

$$a = \frac{F_{rot,r,x}}{M}.$$

Now in order that B remains on its position, it follows that

$$l'' = a. \tag{27}$$

Simple calculation shows that (27) leads to

$$F_{rot,r,x} = \frac{I_B - I_C}{I_B} F_{rot,f,x} \tag{28}$$

which denotes the lateral force $F_{rot,r,x}$ at the rear tyre in case there is a lateral force $F_{rot,f,x}$ causing rotation at the front tyre.

4.6 Acceleration

The last movement component of the presented model is given by the longitudinal acceleration dv/dt of the car body. Recall the basic car force model as depicted in Fig. 3 (a). Also recall the definitions of the traction forces $\boldsymbol{F}_{trac,f}$ by (11) and $\boldsymbol{F}_{trac,r}$ by (12).

Longitudinal acceleration mainly is created by the car engine and the tyre breaks. An engine puts a *torque* $T_{e,r}$ to the rear axle, which itself forwards this torque to the rear tyre (in this model there is only one central tyre). This torque then results in an angular acceleration of the rear tyre, and a traction force $F_{trac,r,y}$ between the rear tyre and the ground. This traction force then accelerates the car body (and the car body then accelerates the front tyre). Alternatively, the tyre breaks would cause negative torques $T_{e,r}$ at the rear and $T_{e,f}$ at the front tyre. In the presented model, there is no slip ratio between the tyres and the ground. If a slip ratio is desired, then the model must be augmented with a corresponding slip factor.

In order to establish the corresponding equations, we define the radius of a tyre by R_w, and the tyre inertia by I_w. Note that the engine torque is split into two parts, a $T_{trac,r,y}$ creating the traction force $F_{trac,r,y}$, and the second part $T_{tot,r}$ actually accelerating the rear tyre (Fig. 8). By noting (7) we get:

$$T_{tot,r} = T_{e,r} - F_{trac,r,y}\, R_w. \tag{29}$$

At the front tyre things get more complicated. The traction force at the front tyre actually is the sum

$$\begin{aligned} \boldsymbol{F}_{trac,f} &= \boldsymbol{F}_{cp,f} + \boldsymbol{F}_{rot,f} + \\ &\quad + \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} 0 \\ F_{acc,f} \end{pmatrix}. \end{aligned} \tag{30}$$

Here the force $F_{acc,f}$ is used similarly to $F_{trac,r,y}$ in (29):

$$T_{tot,f} = T_{e,f} - F_{acc,f}\, R_w. \tag{31}$$

By using (21), (26) and (30), the y-coordinate of $\boldsymbol{F}_{trac,f}$ is given by

$$\begin{aligned} F_{trac,f,y} &= F_{cp,f,y} - \\ &\quad - f_r \left(\tan\beta \frac{dv}{dt} + \frac{v}{\cos^2\beta} \frac{d\beta}{dt} \right) + \\ &\quad + F_{acc,f} \cos\beta, \end{aligned} \tag{32}$$

here setting

$$F_{cp,f,y} = \frac{-Mv^2 \tan^2\beta}{2L}$$

and

$$f_r := \frac{I_B \tan\beta}{L^2}. \tag{33}$$

For the car body, the longitudinal traction forces sum up and result in the total longitudinal acceleration force F_{tot} on the car body:

$$F_{tot} = F_{trac,r,y} + F_{trac,f,y}. \tag{34}$$

Note that in (34), the term $F_{trac,f,y}$ can immediately be replaced by the right hand side of (32). Equ. (34) is also a good place for adding additional terms for various drag forces, which is not done here. What remains is the coupling between the rotational acceleration of the tyres and the car acceleration, which due to (2), (6), (7), (8), and (18) result in

$$\frac{dv}{dt} = \frac{F_{tot}}{M} = \frac{R_w T_{tot,r}}{I_w}, \tag{35}$$

and

$$\frac{dv}{dt} = \frac{F_{tot}}{M} = \frac{|\boldsymbol{v}_B|}{|\boldsymbol{v}_A|} \frac{R_w T_{tot,f}}{I_w} = \cos\beta \frac{R_w T_{tot,f}}{I_w}. \tag{36}$$

The meaning of the system of linear equations (29) to (36) is this: if a torque is put onto the rear (front) tyre, this torque is split into the parts accelerating the rear (front) tyre, the front (rear) tyre and the car body. The velocity of the car mass is equal to the rotational velocity of each tyre. However, the velocity of the front tyre must be multiplied by a term which corrects the different movement radii. Additionally, if a slip ratio different to 1 is desired, it can be added to (35) and (36).

The above system of linear equations (29) to (36) can be solved analytically. The solution for F_{tot} is given by

$$\begin{aligned} F_{tot} &= \frac{MR_w^2 \left(F_{cp,f,y} - \frac{f_r v}{\cos^2\beta} \frac{d\beta}{dt} \right)}{2\, I_w + (M + f_r \tan\beta) R_w^2} + \\ &\quad + \frac{MR_w \left(\cos\beta\, T_{e,f} + T_{e,r} \right)}{2\, I_w + (M + f_r \tan\beta) R_w^2} \end{aligned} \tag{37}$$

The first summand of (37) denotes the rotational and centripetal forces, while the second part denotes the engine and

Parameter	Value	Unit	Explanation
W	2	m	Car width
L	4	m	Car length
M	1500	kg	Car mass
R_w	0.33	m	Radius of tyre
I_w	2×4.1	kg m^2	Inertia of tyre

Table 1: General model parameters.

break torques. Furthermore, from (37), the other unknowns $T_{tot,r}$, $T_{tot,f}$, $F_{trac,r,y}$, and $F_{acc,f}$ can easily be derived using equations (29) to (36). Although (37) looks quite complicated, for $\beta \approx 0$ it changes into a much simpler form, because in this case $\cos\beta \approx 1$ and $\tan\beta \approx f_r \approx F_{cp,f,y} \approx 0$:

$$F_{tot} \approx \frac{MR_w\,(T_{e,r} + T_{e,f})}{2I_w + MR_w^2}.$$

The M in the numerator is of course the car mass, while the denominator represents the inertia of both tyres plus the car mass. Additionally, the numerator just sums up the engine and break torques of the rear and front tyres.

5. TRACTION

The traction forces depicted in Fig. 3 (a) are exchanged between the tyres and the ground. However, there is a limit F_{max} to the amount of force that can be exchanged. If the traction force on one of the tyre exceeds this bound, then the tyre looses grip and starts sliding. It follows that the model must continuously check whether the traction forces exceed the maximum. In such a case, the model then must switch to an appropriate sliding model, which is not treated in this work, but for instance in [10].

Following [6], F_{max} is determined by the static friction factor μ_s in the following way. If F denotes the force that presses down the tyre to the ground, then

$$F_{max} = \mu_s\, F.$$

Here, F is the amount of weight force on one tyre, i.e., in the presented model $F = 9.81M/2$. Values for μ_s are, for instance, $\mu_s = 1.0$ for rubber on dry concrete ground, or $\mu_s = 0.3$ for rubber on wet concrete ground. If the car starts sliding, then the force exchanged between tyre and ground is asssumed to be constant: $F_{trac} = \mu_k\, F$. Here μ_k denotes the kinetic friction factor which, for instance is $\mu_k = 0.8$ for rubber on dry concrete ground, or $\mu_k = 0.25$ for rubber on wet concrete ground. Other values for μ_s and μ_k can be found, for instance, in [6, 1].

6. MODEL SOLUTION

The previous sections have presented numerous equations, their integration into a closed model is described in this section.

For solving the model, we now define a set of parameters and input variables which define the state of the model and user input. The set of basic model parameters is shown in Tab. 1.

The *current model state* is given by the velocity $\boldsymbol{v}_B = (0, v)^T$ of point B. *User input* is given by the steering angle β and the engine/break torques $T_{e,r}$ and $T_{e,f}$ as defined in Section 4.6.

```
InitializeModel();
double lasttime = now();
while( true ) {
    Input input;
    GetUserInput( input );
    double time = now();
    ComputeModel( input, time - lasttime );
    RenderTheScene();
    lasttime = time;
}
```

Table 2: The game loop.

The game loop, which is executed continuously throughout the game, is shown in Tab. 2. At first the user input is queried from either keyboard or joystick. Then the function responsible for computing the model is called. Its input parameters are the user input, here a structure holding β, $T_{e,r}$ and $T_{e,f}$, and the time since its last call.

The code for computing the model itself is shown in Tab. 3. The function is called after some time dt has gone by. The function then computes the model behaviour for the last dt seconds, i.e., for the time interval that has passed by. The structure `oldinput` stores the user input that was recorded at the start of this interval, while the structure `input` holds the user input at the end of the interval. Engine and breaking forces either can be taken from `oldinput` and can be kept constant throughout the interval, or the average between the engine and breaking forces of `oldinput` and `input` can be used. In Tab. 3 the first alternative is chosen.

7. EXPERIMENTAL RESULTS

The pseudocode from Tab. 3 has been implemented by using Mathematica 5.2.[7] The implementation has been used for running a number of experiments. The purpose of the experiments was to test whether the presented model can be implemented as described in Tab. 3 and yields numerical results that make sense. Also the experiments should show the magnitude of the observed traction forces, including centripetal forces (which are mainly used in other models), but also the additional forces as described in this paper, and furthermore investigate the usefulness of the given explicit formulas for explaining numerical results.

In the first set of experiments, the car is only accelerated without steering, i.e., $\beta = 0$ and $T_{e,r} > 0$. Furthermore, it is assumed that acceleration on the front tyre is only done via the brakes, which are not used in the presented experiments. Thus, throughout all experiments, $T_{e,f} = 0$. Also, the time difference was set to $dt = 10\,$ms. The traction forces and the force on the main car body for acceleration only can be seen in Fig. 9. The figure additionally shows the maximum allowed forces for dry (upper horizontal thin line) and wet (lower horizontal thin line) ground, as defined in Section 5. If the traction force at any tyre exceeds this bound then the car starts sliding. It can be seen that almost all of the

[7] http://www.wolfram.com/

```
Input oldinput;
double v = 0;
ComputeModel( Input input, double dt ) {
    if( dt > 0 ) {
        β = oldinput.β
        dβ = input.β − oldinput.β
        T_{e,f} = oldinput.T_{e,f}
        T_{e,r} = oldinput.T_{e,r}

        Compute ω from (23)
        if( β ≠ 0 ) {
            Compute F_{cp,f} from (21)
            Compute F_{cp,r} from (22)
        } else {
            F_{cp,f} = (0,0)^T and F_{cp,r} = (0,0)^T
        }
        Compute f_r from (33)
        Compute F_{tot} from (37)
        a = F_{tot}/M
        Compute α from (24)
        Compute F_{trac,r,y}, F_{acc,f}, T_{tot,r}, and T_{tot,f}
            from (29) to (36)
        Compute F_{rot,f} from (25) and (26)
        Compute F_{rot,r} from (28)

        F_{trac,r} = (0, F_{trac,r,y})^T + F_{cp,r} + F_{rot,r}
        Compute F_{trac,f} from (30)

        F_{max} = 9.81 M/2
        if( |F_{trac,r}| ≤ F_{max} && |F_{trac,f}| ≤ F_{max} ) {
            Advance the car by ds = v dt + a dt^2/2
            Rotate the car by dθ = ω dt + α dt^2/2
            v = v + a dt
        } else {
            Switch to sliding model
        }
    }
    oldinput = input;
}
```

Table 3: Computing the model.

resulting body acceleration F_{tot} is due to the rear traction force $F_{trac,r,y}$. The front traction force $F_{trac,f,y}$ here only is responsible for accelerating the front tyre.

In the next experiments it is assumed that the car is travelling with some velocity $v > 0$ and a nonzero steering angle $\beta > 0$. However, there is no engine or break torque on the tyres. At first, the steering angle β was set to 1 degree, and the steering angle change $d\beta/dt$ was set to $d\beta/dt = 0, 1, 5$, and 10 deg/s. The resulting front traction force $|\boldsymbol{F}_{trac,f}|$ is shown in Fig. 10. Here it must be noted that if $\beta \neq 0$ and $d\beta/dt = T_{e,r} = T_{e,f} = 0$, then the car body acceleration force F_{tot} is quite small, but due to the y-coordinate of (21) not zero. It follows that dv/dt is also quite small, resulting due to (24) in a *small* rotational force $\boldsymbol{F}_{rot,f}$ and $\boldsymbol{F}_{rot,r}$. Thus, the observed traction forces are mainly caused by the centripetal forces, i.e., $\boldsymbol{F}_{trac,r} \approx \boldsymbol{F}_{cp,r}$ and $\boldsymbol{F}_{trac,f} \approx \boldsymbol{F}_{cp,f}$. If however the steering angle changes, then it can be seen that the traction forces quickly rise. This additional growth

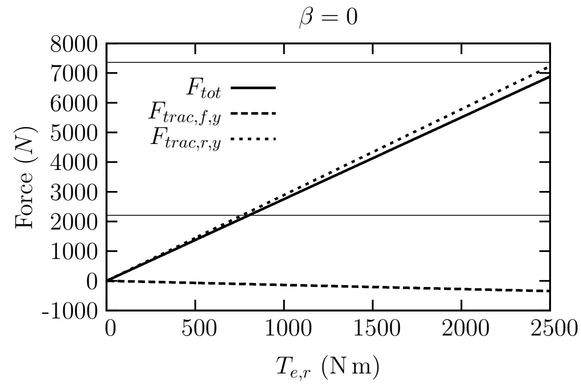

Figure 9: Forces caused by acceleration only ($\beta = 0, d\beta/dt = 0, T_{e,f} = 0$).

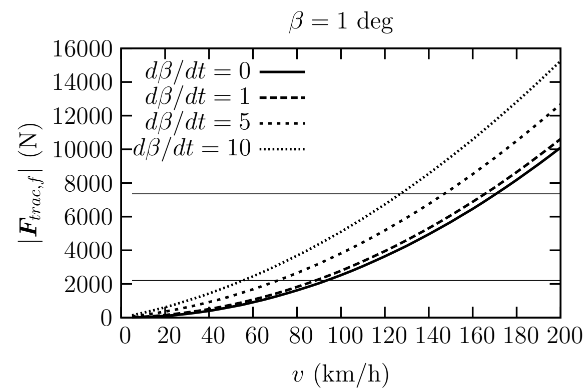

Figure 10: Front traction force $|\boldsymbol{F}_{trac,f}|$ caused by steering only ($\beta = 1\,\mathrm{deg}, T_{e,r} = 0, T_{e,f} = 0$). For $d\beta/dt = 0$, $|\boldsymbol{F}_{trac,f}|$ is almost equal to $|\boldsymbol{F}_{cp,f}|$.

is caused by the rotational forces $\boldsymbol{F}_{rot,r}$ and $\boldsymbol{F}_{rot,f}$, which additionally are shown in Fig. 11.

The same experiments have been repeated, this time setting $\beta = 3\,\mathrm{deg}$. The results are shown in Figs. 12 and 13. Especially Fig. 12 shows that when the steering angle is that large then the traction forces soon become too large and steering is impossible.

The final experiments investigate which values for β and $d\beta/dt$ do not cause sliding when driving with a certain velocity v. In Fig. 14 the maximal allowed steering angle β is shown for dry and wet ground. The thin horizontal lines are drawn at $\beta = 1\,\mathrm{deg}$ and $\beta = 3\,\mathrm{deg}$. The results may, for instance, be used to implement a virtual steering assistance system which prohibits sliding by automatically reducing β, in case the traction forces grow too large.

For $\beta = 1\,\mathrm{deg}$ and $\beta = 3\,\mathrm{deg}$ it has then been investigated how fast the driver may change the steering angle without causing the car to slide. The results are shown in Fig. 15, again for dry and wet ground. The lines end at the points where the traction force is already too large only because of $\beta \neq 0$ (compare to the intersections of the thick lines and the thin horizontal lines in Fig. 14).

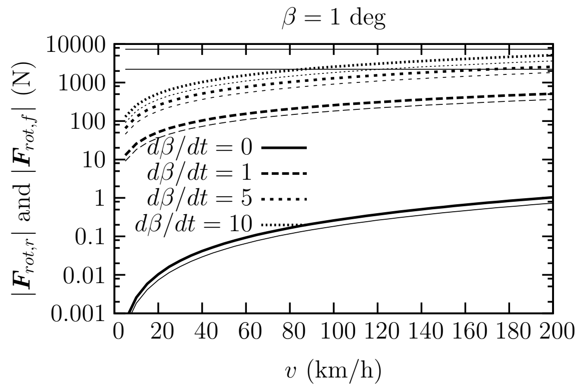

Figure 11: Rotational forces $|\boldsymbol{F}_{rot,f}|$ (thick lines) and $|\boldsymbol{F}_{rot,r}|$ (thin lines) caused by steering only ($\beta = 1$ deg, $T_{e,r} = 0, T_{e,f} = 0$).

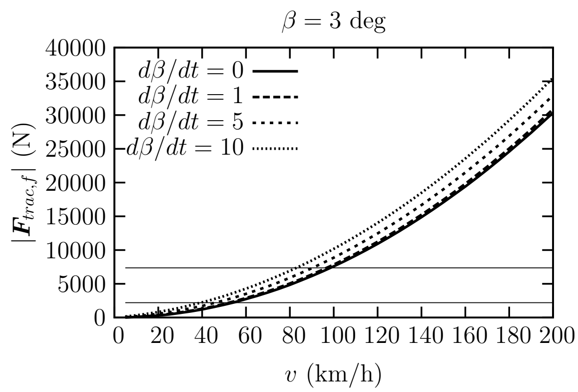

Figure 12: Front traction force $|\boldsymbol{F}_{trac,f}|$ caused by steering only ($\beta = 3\,\mathbf{deg}, T_{e,r} = 0, T_{e,f} = 0$). For $d\beta/dt = 0$, $|\boldsymbol{F}_{trac,f}|$ is almost equal to $|\boldsymbol{F}_{cp,f}|$.

8. CONCLUSIONS

In this paper a detailed car physics model for Ackermann steering is presented. The model consists of closed formulas which derive all necessary forces and thus accelerations for simulating Ackermann steering. In particular, centripetal forces, rotational forces and acceleration forces are described. An interesting result is given for rotational forces at the rear tyre. Furthermore, the identity of a component of the centripetal force at the front tyre, and the force being responsible for rotating the car about its rear tyre is shown. It is then demonstrated how to use the model in a game loop. Finally, an implementation of this loop is used for carrying out a number of numerical experiments.

The model does not include things like weight transfer, tyre slip or engine power. These have been investigated thoroughly in previous papers and can be included into the model easily.

Future work will focus on sliding models, which will model the case when only the front tyres, only the rear tyres, or all tyres loose traction.

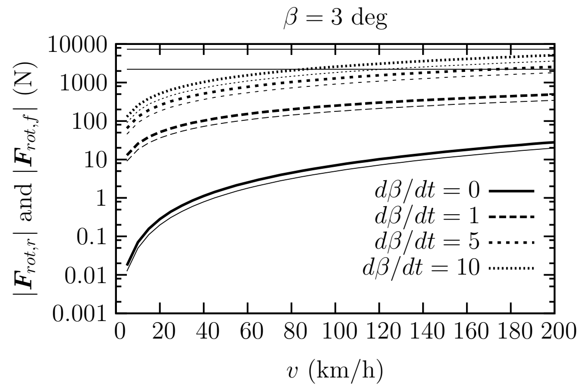

Figure 13: Rotational forces $|\boldsymbol{F}_{rot,f}|$ (thick lines) and $|\boldsymbol{F}_{rot,r}|$ (thin lines) caused by steering only ($\beta = 3$ deg, $T_{e,r} = 0, T_{e,f} = 0$).

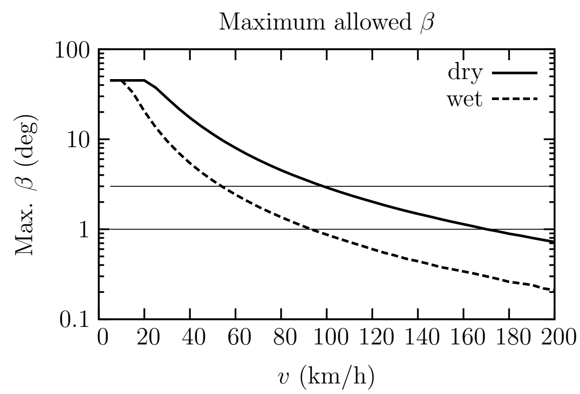

Figure 14: Maximum allowed β (deg) for front traction force $|\boldsymbol{F}_{trac,f}|$ caused by steering only ($d\beta/dt = 0, T_{e,r} = 0, T_{e,f} = 0$).

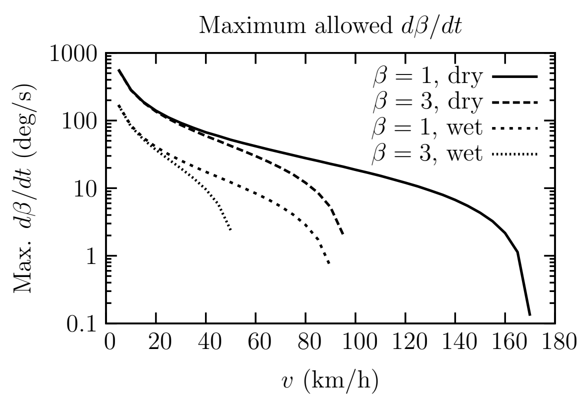

Figure 15: Maximum allowed $d\beta/dt$ (deg/s) for front traction force $|\boldsymbol{F}_{trac,f}|$ caused by steering only ($T_{e,r} = 0, T_{e,f} = 0$).

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